Basic matrix concepts

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Motivation

To see a few of the amazing things matrices allow us to do, check out this great YouTube video! Sneak peek of topics so you actually click on the video: modelling predator-prey relationships (graphs of wolf and rabbit population over time), modelling how a virus spreads through a population, solving crime cases, representing images on a computer through a bitmap (if you're reading this right now, this is highly relevant), blurring and rotating and causing other such transformations to images... if any of that interests you at all, you should watch the video!

1 Definition

A matrix is a rectangular list of numbers. We say rectangular because it has dimensions $m \times n$, like a rectangle. m is the number of rows and n is the number of columns. For example, this is a 2×3 matrix with 6 elements.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

If $m = n$, we have a square matrix.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lastly, we can denote any element of a matrix by its row and column. So for M, we may say the 6 is element $m_{2,3}$ because it is in row 2 and column 3.

2 Addition and subtraction

Addition between matrices is defined only when they have the same dimensions. For example, we cannot perform $M + I$ because their dimensions are not the same. If we let

$$N = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -6 & -4 \end{bmatrix}$$

Then

$$M + N = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ 5 & -6 & -4 \end{bmatrix} = \begin{bmatrix} 1\text{+}2 & 2\text{+}1 & 3\text{+}3 \\ 4\text{+}5 & 5\text{+(-6)} & 6\text{+(-4)} \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 9 & -1 & 2 \end{bmatrix}$$

We call this element-wise addition. For subtraction, we must once again confirm that the dimensions of the two matrices are the same. Here is an example of subtraction.

$$M - N = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ 5 & -6 & -4 \end{bmatrix} = \begin{bmatrix} 1\text{-}2 & 2\text{-}1 & 3\text{-}3 \\ 4\text{-}5 & 5\text{-(-6)} & 6\text{-(-4)} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 11 & 10 \end{bmatrix}$$

3 Scalar multiplication

We can think of a scalar as a coefficient. Scalar multiplication also occurs element-wise, like so:

$$-3N = -3\begin{bmatrix} 2 & 1 & 3 \\ 5 & -6 & -4 \end{bmatrix} = \begin{bmatrix} \text{(-3)}\cdot 2 & \text{(-3)}\cdot 1 & \text{(-3)}\cdot 3 \\ \text{(-3)}\cdot 5 & \text{(-3)}\cdot(-6) & \text{(-3)}\cdot(-4) \end{bmatrix} = \begin{bmatrix} -6 & -3 & -9 \\ -15 & 18 & 12 \end{bmatrix}$$

We can now define $M - N$ as $M + (-1)N$.

4 Matrix multiplication

Multiplication of two matrices AB, in this order, is defined only when the number of columns of A equals the number of rows of B. So A must have dimensions $m \times n$, and B must have dimensions $n \times p$. The resulting matrix has dimensions $m \times p$, the number of rows of A by the number of columns of B. If we write the dimensions of both matrices like so: $(m \times n)(n \times p)$, we can note that the inner two numbers must be equal for multiplication to occur, and the resulting matrix has dimensions $m \times p$, the outer two numbers. Let's transpose, or flip the dimensions, rows, and columns of N, and make all its elements positive[1]. We'll call this new matrix O.

$$O = \begin{bmatrix} 2 & 5 \\ 1 & 6 \\ 3 & 4 \end{bmatrix}$$

We can now do the multiplication MO. M has dimensions 2×3, and O has dimensions 3×2. If we write $(2 \times 3)(3 \times 2)$, the inner two numbers are the same, so we can do the multiplication, and the resulting matrix will have dimensions 2×2, the outer two numbers. Without writing that out explicitly, we can note that columns of M = rows of N = 3, so multiplication is defined, and the resulting dimensions are (rows of M × columns of N) = (2×2). We first calculate the top left element of MO, which I denote $mo_{1,1}$.

$$MO = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 2 \cdot 1 + 3 \cdot 3 & ? \\ ? & ? \end{bmatrix}$$

For $mo_{1,1}$, we work only with row 1 of M and column 1 of O. We sum the product of the first elements of the row and column, the product of the second elements, and the product of the third and final

[1] It'd be a good exercise for you to do the multiplication with the negative signs.

elements. This is why we have the $(m \times n)(n \times p)$ condition! Without this condition, we wouldn't be able to match up the elements like we just did. And since we're "multiplying" (it's actually a dot product) 2 rows of M by 2 columns of O, of course we end up doing this dot process $2 \cdot 2 = 4$ times and put the results into a 2×2 grid. Here's the rest of the multiplication. If this doesn't make sense yet, you should watch this Khan Academy video for a visual explanation.

$$MO = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 3 & 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 4 \\ 4 \cdot 2 + 5 \cdot 1 + 6 \cdot 3 & 4 \cdot 5 + 5 \cdot 6 + 6 \cdot 4 \end{bmatrix} = \begin{bmatrix} 13 & 29 \\ 31 & 74 \end{bmatrix}$$

Do you remember I from the first page? Let's do $I(MO)$ and MOI. We don't need to check explicitly that $I(MO)$ and MOI are defined since matrix multiplication is always defined for matrices with the same dimensions[2].

$$I(MO) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 29 \\ 31 & 74 \end{bmatrix} = \begin{bmatrix} 1 \cdot 13 + 0 \cdot 31 & 1 \cdot 29 + 0 \cdot 74 \\ 0 \cdot 13 + 1 \cdot 31 & 0 \cdot 29 + 1 \cdot 74 \end{bmatrix} = \begin{bmatrix} 13 & 29 \\ 31 & 74 \end{bmatrix}$$

$$MOI = \begin{bmatrix} 13 & 29 \\ 31 & 74 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 \cdot 1 + 29 \cdot 0 & 13 \cdot 0 + 29 \cdot 1 \\ 31 \cdot 1 + 74 \cdot 0 & 31 \cdot 0 + 74 \cdot 1 \end{bmatrix} = \begin{bmatrix} 13 & 29 \\ 31 & 74 \end{bmatrix}$$

So I, a square matrix with 1's across the diagonal from the top left element to the bottom right element and 0's everywhere else (if $m = n$, $i_{mn} = 1$, else $i_{mn} = 0$) is the identity matrix, which doesn't do anything during multiplication! We can multiply it on the left or on the right.

Does this fact imply that matrix multiplication is commutative ($AB \stackrel{?}{=} BA$)? Well, we can conclude that I is commutative[3], but $AB \neq BA$ in general. We could prove this by finding a counterexample, but we don't really need to. Recall the $(m \times n)(n \times p)$ rule for matrix multiplication. Let A have dimensions $m \times n$ and let B have dimensions $n \times p$ so that AB is defined. Then BA looks like $(n \times p)(m \times n)$. If $p \neq m$, then BA isn't even defined[4]! In general, even if BA is defined, it often results in a matrix that's different from the one we would get from AB, if AB is defined.

But, matrix multiplication is associative, meaning $(AB)C = A(BC)$, assuming multiplication is defined. We can think of this as meaning that we're safe as long as we do multiplication from left to right (for example, CBA is not associative because it multiplies from right to left).

5 Determinant of a 2x2 matrix

The determinant is defined only for square matrices. The determinant of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $ad - bc$. Multiply the elements of the main diagonal (top left to bottom right) and subtract the product of the elements of the off-diagonal (top right to bottom left). For example, let

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$\det(P) = |P| = 1(4) - 2(3) = -2.$

[2] Why?

[3] Why?

[4] Recall MO from the previous page. Is OM defined? If so, what are the resulting dimensions? How about MN?

Conclusion

That's all you need to know for the ACT. Well, there's one more important thing - TI calculators can do all basic matrix operations discussed in this document. However, you might not have read to here if I said this up front, and I've seen several matrix multiplication questions in which a calculator would not help and might actually be actively detrimental. If you want to learn about a method for finding the determinant of higher-order matrices or what it's used for, which won't be tested, you should visit this Math is Fun article. This is actually where I first learned about matrix multiplication and determinants! Please visit this Khan Academy page to do some practice problems - it's the only way to learn.

Bonus

Consider augmented matrix P_1. An augmented matrix is just a matrix with one extra column representing the numbers the equations are equal to.

$$P_1 = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right]$$

This represents the following system of equations.

$$\begin{cases} x + 2y = 5 \\ 3x + 4y = 6 \end{cases} \tag{1}$$

This system has 1 solution. It has 1 solution no matter what the equations are equal to; that is, the numbers in the third column don't actually matter in this case. Graphically, the two equations are two lines, and we know a solution to the system is the pair of coordinates at which the lines intersect. We also know that the two lines have unequal slopes. You should confirm this[5]. Infinitely long lines with unequal slopes must have precisely one intersection or precisely one solution.
Now consider augmented matrix P_2.

$$P_2 = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 6 & 6 \end{array}\right]$$

$$\begin{cases} x + 2y = 5 \\ 3x + 6y = 6 \end{cases} \tag{2}$$

It should be clear, by algebra[6] and/or graphing, that this system has no solutions. There is another way of arriving at the same conclusion.
$\det(P_1) = 1 \cdot 4 - 2 \cdot 3 = -2 \neq 0 \implies$ one solution
$\det(P_2) = 0 \implies$ no solutions[7]
A square matrix with determinant 0 is known as a singular matrix.

[5]Solve for the slope and note that the slope does not depend on the values in the third column.
[6]Hint: equation $2 = 3 \cdot$ equation 1 but $6 \neq 3 \cdot 5$
[7]Is it always true that whenever the determinant of a 2x2 matrix is 0, there are no solutions?