# Advanced Math Concepts for the ACT®

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Published August 10, 2022, last updated May 30, 2023

This document covers advanced ACT Math concepts. It is intended for high scorers who want a perfect score. You should focus on basic concepts, which are more common, if you don't have at least a 30. There is some advanced notation (e.g. $\sum$, $\cap$, $\cup$, $\lfloor\,\rfloor$), but you can look them up and Google for definitions.

I made this document by looking at past ACT® tests. This document has all "advanced" concepts from Forms E26 (June 2022), Z08 (April 2022), E25 (April 2022), E23 (December 2021), and D06 (June 2021) up to question 31.



## Contents

# 1 Counting

See [Wei22b] and MathIsFun for more details.

## 1.1 Permutations

With repetition, the number of ordered arrangements [1] of $n$ objects into $k$ slots is given by $n^k$. Without repetition, the number of ordered arrangements [2] of $n$ objects into $k$ slots is given by (1.1.1).

$$^nP_k = \frac{n!}{(n-k)!} \tag{1.1.1}$$

## 1.2 Combinations

Without repetition, the number of (unordered) ways to choose $k$ objects out of $n$ objects [3] is given by (1.2.1).

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.2.1}$$

[1] For example, the number of 3-digit numeric (base-10) passwords ($n = 10$, $k = 3$)
[2] For example, the number of 3-digit numeric (base-10) passwords with distinct digits ($n = 10$, $k = 3$)
[3] For example, the number of ways to choose a team of 3 runners from 10 people ($n = 10$, $k = 3$)

# 2 Probability

## 2.1 Expected value

Expected value[4], denoted by **E**, can be thought of as a weighted average. It operates on random variables.

A random variable takes on a random value out of a set of values with specified probabilities. For example, let $D$ be the number that faces up when a fair six-sided die is rolled. Then $D$ is a random variable that takes values $\{1,2,3,4,5,6\}$ with probabilities $\{\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$. See mean_std.ipynb and helpers for more examples.

**Definition 2.1.1: Expected value**

Let $X$ be a discrete random variable that takes values $x_i$ with probabilities $p_i$. Its expected value is defined as

$$\mathbf{E}X = \sum_i x_i p(x_i) \tag{2.1.1}$$

**Problem 2.1.1: Gambling**

Suppose I offer you the following game, in which you will roll a fair six-sided die. Would you play the game?

- If a 6 is rolled, you win \$6.
- If a 5 is rolled, you win \$3.
- If a 4 is rolled, you don't win or lose anything.
- Otherwise, you lose \$4.

*Solution.* Let $X$ = money gained or lost. Then $\mathbf{E}X = \sum_i x_i p(x_i) = 6\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 0\left(\frac{1}{6}\right) - 4\left(\frac{3}{6}\right) = -0.5$. For a sufficiently large number of games, there is an average loss per game of \$0.50, so we don't want to play. ■

## 2.2 Inclusion-exclusion principle

Note that $\cup$ denotes set union (similar to "or"). $\cap$ denotes set intersection (similar to "and"). [5]

The inclusion-exclusion principle [6] for two events $A$ and $B$ given in (2.2.1) holds whether $A$ and $B$ are mutually exclusive or not. When they are mutually exclusive, $\mathbf{P}(A\cap B) = 0$, so the probability that $A$ or $B$ occurs is just $\mathbf{P}(A)+\mathbf{P}(B)$. When they are not mutually exclusive, then $\mathbf{P}(A)+\mathbf{P}(B)$ doubly counts the intersection, so the intersection needs to be subtracted. See Figure 1.

$$\mathbf{P}(A\cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A\cap B) \tag{2.2.1}$$

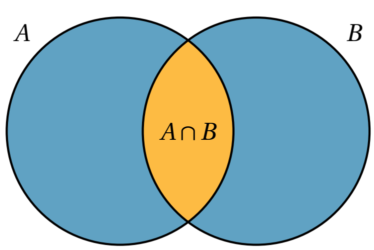



Figure 1: Inclusion exclusion principle

[4] More formally, expectation
[5] In Figure 1, $A\cup B$ includes the blue and orange sections. $A\cap B$ is just the orange
[6] See this page for inclusion-example examples

# 3 Logarithms

See [Wei21b]. Topics covered are logarithm rules, practice problems, and solutions.

# 4 Matrices

See [Wei21a] for much more details. Topics covered are the definitions of addition and subtraction, scalar multiplication, matrix multiplication, and the determinant of a $2 \times 2$ matrix.

## 4.1 Multiplication

In the example shown in (4.1.1), the two matrices can be multiplied because their dimensions are of the form $m \times n$ and $n \times p$ in this order. The resulting matrix has dimensions $m \times p$. Note that matrix multiplication is not commutative ($AB \neq BA$, usually).

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} ag + bh \\ cg + dh \\ eg + fh \end{pmatrix} \tag{4.1.1}$$

## 4.2 Determinant

The determinant is defined for square matrices only. On the ACT®, if you are asked to calculate a determinant, it will be the determinant of a $2 \times 2$ matrix. Here is the formula.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{4.2.1}$$

# 5 Geometry

## 5.1 2D Point rotation

If it's hard to remember (5.1.1), visualize a simple case that should be obvious: $(10,1) \xrightarrow{90^\circ\text{CCW}} (-1,10)$. Also, note that (5.1.2) is equal to three applications of (5.1.1), and similar for (5.1.3).

$$(a,b) \xrightarrow{90^\circ\text{CCW}} (-b,a) \tag{5.1.1}$$

$$(a,b) \xrightarrow{90^\circ\text{CW}} (b,-a) \tag{5.1.2}$$

$$(a,b) \xleftrightarrow{180^\circ} (-a,-b) \tag{5.1.3}$$

## 5.2 2D Point reflection

$$(a,b) \xrightarrow{\text{x axis}} (a,-b) \tag{5.2.1}$$

$$(a,b) \xrightarrow{\text{y axis}} (-a,b) \tag{5.2.2}$$

## 5.3 Vectors

Addition, subtraction, and scalar multiplication of vectors follow the definitions given in Section 4.

Graphical definitions of vector addition, subtraction, and scalar multiplication are also sometimes tested, so we will discuss those using the example vectors $\vec{u} = \langle 1,2 \rangle$ and $\vec{v} = \langle 3,-4 \rangle$. [7]

[7] Note that, following convention, $\hat{\mathbf{i}} = \langle 1,0 \rangle$ and $\hat{\mathbf{j}} = \langle 0,1 \rangle$ so that $\vec{u} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\vec{v} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

### 5.3.1 Addition

The graphical representation of $\vec{u}+\vec{v}$ is shown in Figure 2. Draw $\vec{u}$ with its tail at any position. [8] Then draw $\vec{v}$ with its tail at the tip of $\vec{u}$. $\vec{u}+\vec{v}$ is now the vector that goes from the tail of $\vec{u}$ to the tip of $\vec{v}$.

See Figure 2.

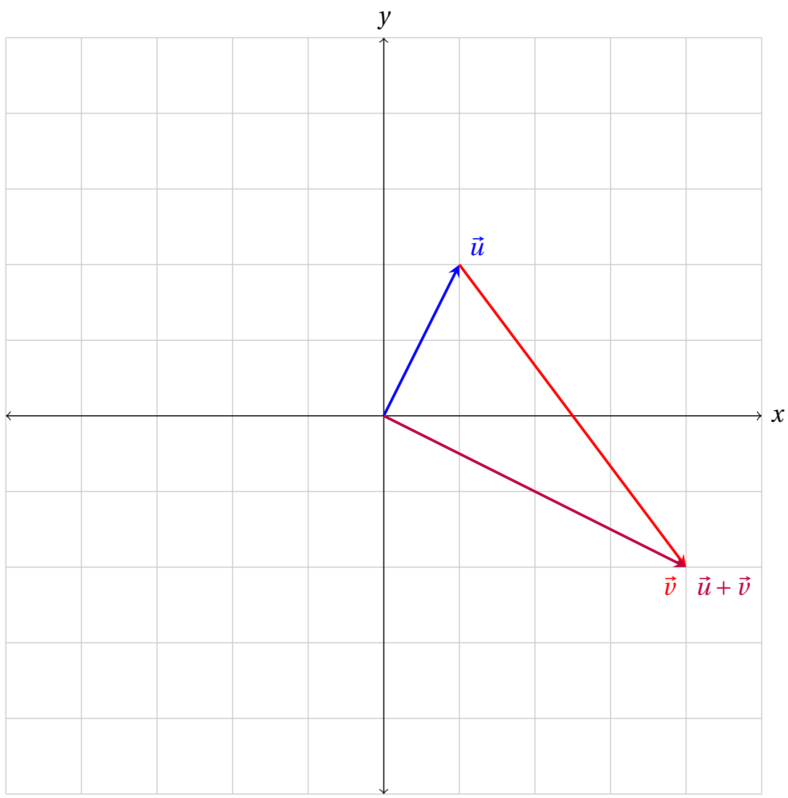



Figure 2: $\vec{u}+\vec{v} = \langle 4, -2\rangle$

### 5.3.2 Scalar multiplication

Graphically, a scalar “scales" a vector. If the scalar is negative, then the vector’s direction is flipped.[9]

See Figure 3.

[8]For ease of calculation, start at the origin. But recall that a vector can be anywhere on the 2D plane as long as its $x$ and $y$ components are the right lengths

[9]See (5.1.3), the rule that describes how a point’s angle changes when its coordinates are negated

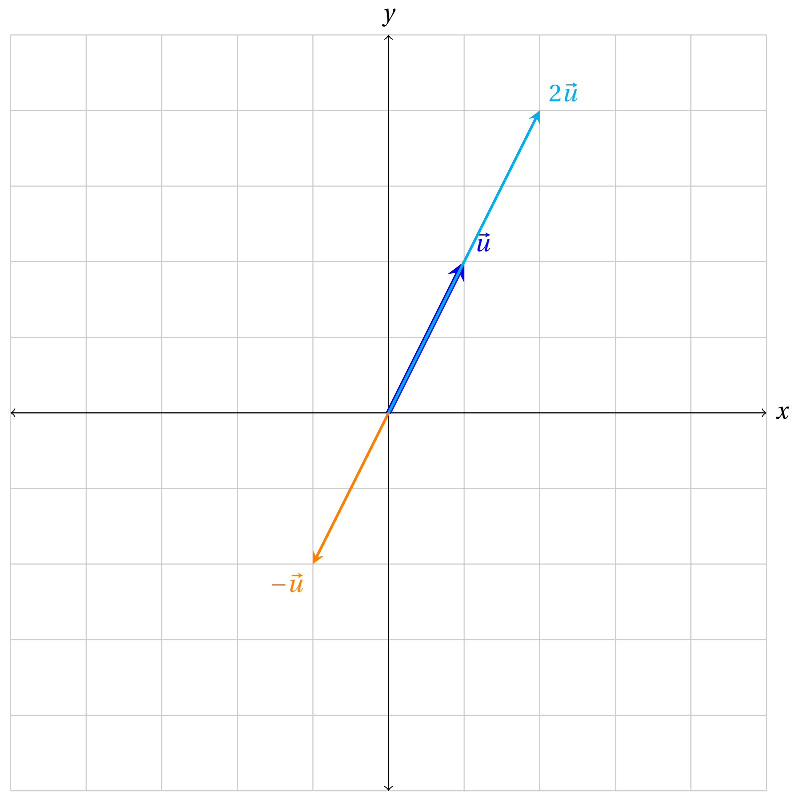



Figure 3: $2\vec{u} = \langle 2,4\rangle$ and $-\vec{u} = \langle -1,-2\rangle$

### 5.3.3 Subtraction

Now, using Section 5.3.2, we can define graphical vector subtraction $\vec{u}-\vec{v}$ as $\vec{u}+(-\vec{v})$. Flip the direction of $\vec{v}$ and follow the graphical addition procedure defined in Section 5.3.1.

See Figure 4.

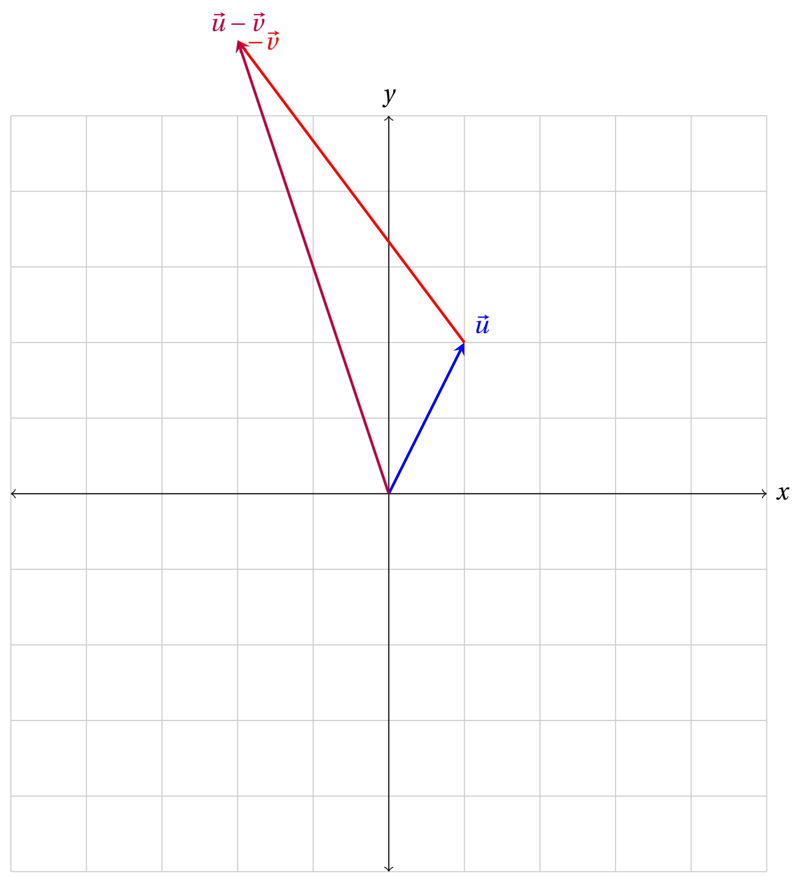



Figure 4: $\vec{u}-\vec{v}=\langle -2,6\rangle$

### 5.3.4 Polar representation

Like points, 2D vectors can also be represented by a radius and an angle, and we can convert from polar representation to Cartesian representation. [10] Let $\vec{A}$ have radius $A$ and make an angle of $\theta$ *with the positive x axis*, as shown in Figure 5. Then by right angle trigonometry, its x and y components (or the components in the direction of $\hat{\mathbf{i}}=\langle 1,0\rangle$ and $\hat{\mathbf{j}}=\langle 0,1\rangle$) are given by $A\cos\theta$ and $A\sin\theta$, respectively. [11]

To convert from Cartesian representation to polar representation, use that $r=\sqrt{A_x^2+A_y^2}$ and the angle between the x and y components is $\arctan(A_y/A_x)$.

A word of caution: be very careful with which axis the angle is measured from. If you always measure $\theta$ from the positive x axis (using angles in the interval $[0,360°)$), then the x component will always be $A\cos\theta$ and the y component will always be $A\sin\theta$.

However, especially in physics, it is not always convenient to measure angles from the positive x axis. In that case, depending on which axis $\theta$ is measured from, the above may not be true, and there may also be issues with sign.

When $\theta$ is not measured from the positive x axis, it is the case that, relative to $\theta$, $A\cos\theta$ gives the magnitude of the adjacent component and $A\sin\theta$ gives the magnitude of the opposite component.

Please practice this concept on Khan Academy. It is impossible to understand through text only.

[10] You generally want to do this when adding or subtracting vectors. You can't directly add vectors that don't have the same angle without converting to component form

[11] Verify this using $\cos\theta=\text{adjacent}/\text{hypotenuse}$ and $\sin\theta=\text{opposite}/\text{hypotenuse}$, where the hypotenuse is the vector $\vec{A}$

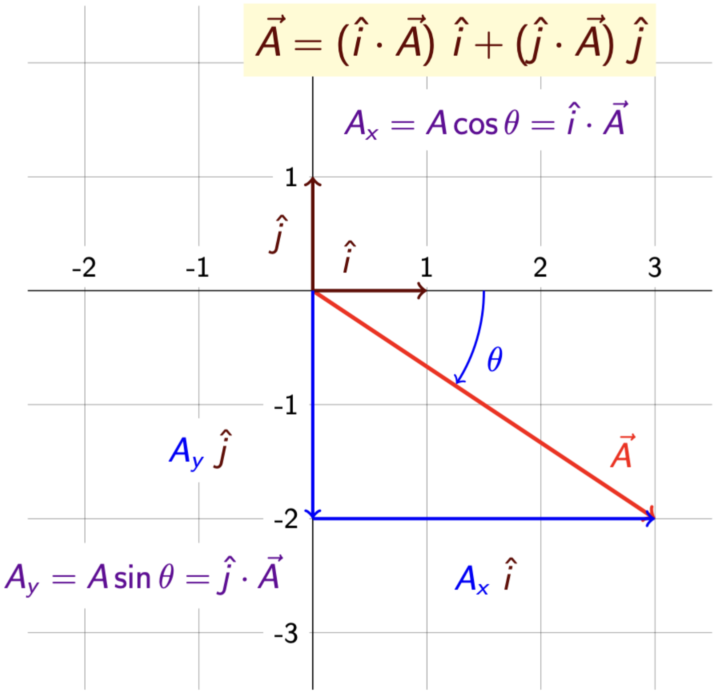



Figure 5: Polar and cartesian representations

### 5.4 Angles

The sum of the interior angles of an $n$-sided polygon is given by $180°(n-2)$, and the sum of the exterior angles is $360°$.

### 5.5 Area and volume formulas

In (5.5.1), the third formula is Heron's formula. In that formula, $s$ is the semiperimeter $(a+b+c)/2$, which is half of the perimeter. You will most likely never need to use this. Also, note that $V_{\text{cone}}$ is $\frac{1}{3}$ of $V_{\text{cylinder}}$ and $V_{\text{rectangular pyramid}}$ is $\frac{1}{3}$ of $V_{\text{rectangular prism}}$.

$$A_{\text{triangle}} = \frac{1}{2}bh = \frac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)} \tag{5.5.1}$$

$$V_{\text{cylinder}} = \pi r^2 h \tag{5.5.2}$$

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h \tag{5.5.3}$$

$$V_{\text{rectangular pyramid}} = \frac{1}{3}Bh = \frac{1}{3}lwh \tag{5.5.4}$$

#### 5.5.1 Unit conversions in higher dimensions

Note that $1\text{m} = 100\text{cm} \not\Rightarrow 1\text{m}^3 = 100\text{cm}^3$.

**Problem 5.5.1:** $\text{m}^3$ **to** $\text{cm}^3$

How many $\text{cm}^3$ are in $1\text{m}^3$?

*Solution.* $1\text{m}^3 = (1\text{m})(1\text{m})(1\text{m}) = (100\text{cm})(100\text{cm})(100\text{cm}) = 100^3\text{cm}^3 = 1000000\text{cm}^3$ ∎

Visualize this by considering a $1\text{m}^3$ cube. Each side length is then 1m. Converting to cm, $V = lwh = (100\text{cm})^3 = 1000000\text{cm}^3$.

### 5.6 Circle properties

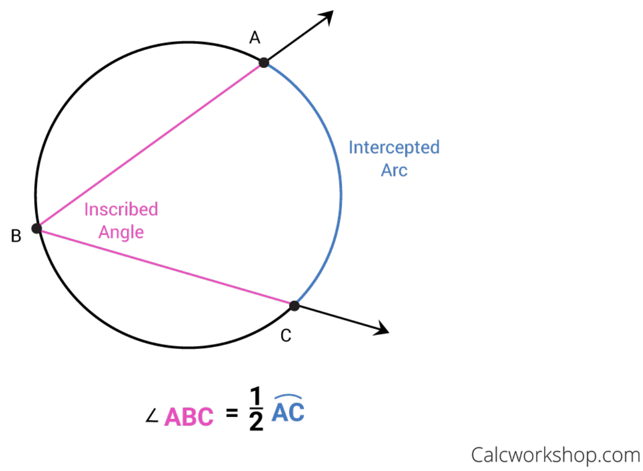



Figure 6: Inscribed angle theorem

### 5.7 Ellipse

The general equation of an ellipse [12] is given in (5.7.1). See (and play around with) this Desmos page to learn about the major/minor axis, the parameters $a$ and $b$, and how to calculate the positions of the foci of the ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{5.7.1}$$

### 5.8 Hyperbola

The general equation of a hyperbola is given in (5.8.1). Generally, the equation/graph of a hyperbola is very rarely tested. When it is tested, it is usually possible to get by just with basic knowledge, but you can see this page for full details.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{5.8.1}$$

## 6 Trigonometry

I hope you already know that in any right triangle, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$. Also (and not necessarily in a right triangle), $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

### 6.1 Graphing

The general form of a basic (sin or cos) trigonometric equation is shown in (6.1.1). See (and play around with) this Desmos page to learn the meanings of the parameters $a$, $b$, $c$, $d$.

$$f(x) = a\sin(b(x+c)) + d \tag{6.1.1}$$

[12] Note that if $a = b$, then this equation becomes the equation of a circle with radius $a$

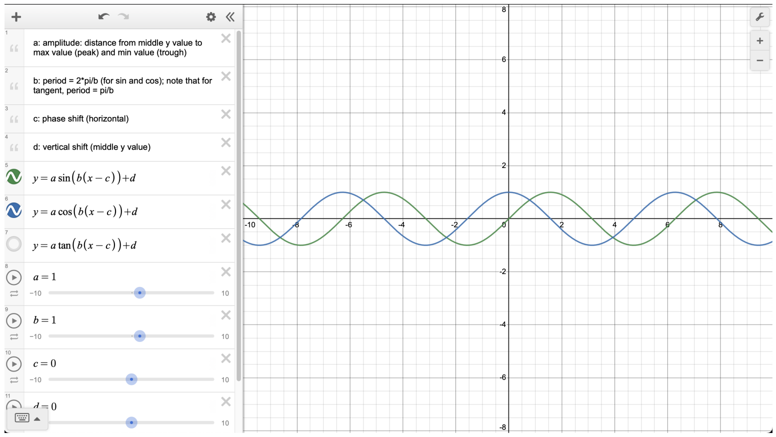



Figure 7: General trig graph

## 6.2 csc, sec, cot

I recommend memorizing (6.2.1) and (6.2.2) using that csc starts with c and goes with sin (which starts with the other letter, s), and similar for sec and cos.

$$\csc\theta = \frac{1}{\sin\theta} \tag{6.2.1}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{6.2.2}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \tag{6.2.3}$$

## 6.3 Law of sines

(6.3.1) and (6.4.1) follow the normal convention of uppercase-lowercase letter pairs. The uppercase letter represents an angle, and the corresponding lowercase letter is the opposite side. Note that these laws apply for all triangles, not just right triangles.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{6.3.1}$$

## 6.4 Law of cosines

Note that the Pythagorean formula is a special case of (6.4.1) in which $C = 90°$ (i.e. in a right triangle).

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{6.4.1}$$

## 6.5 Necessary identities

(6.5.1) comes from applying the Pythagorean formula $a^2 + b^2 = c^2$ to the point at arbitrary $\theta$ on the unit circle, and (6.5.2) and (6.5.3) come from angle sum identities or from reasoning about the point at an arbitrary angle $\theta$ and $-\theta$ on the unit circle. See Figure 8 and [Wei22c].

**Definition 6.5.1: Pythagorean identity**

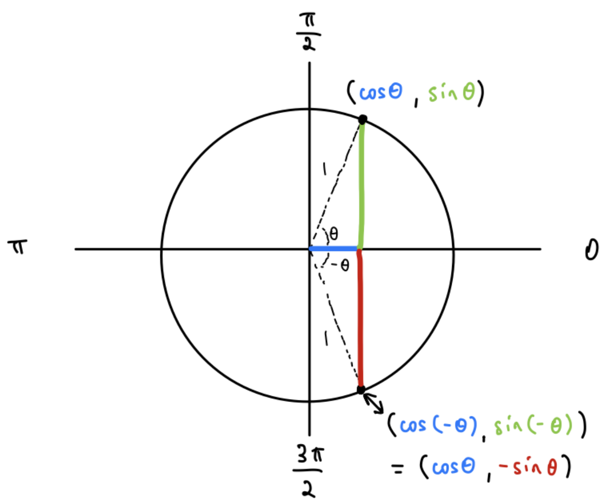
$$\sin^2\theta + \cos^2\theta = 1 \tag{6.5.1}$$

**Definition 6.5.2:** $\sin$ **is an odd function**

$$\sin(-\theta) = -\sin\theta \tag{6.5.2}$$

**Definition 6.5.3:** $\cos$ **is an even function**

$$\cos(-\theta) = \cos\theta \tag{6.5.3}$$



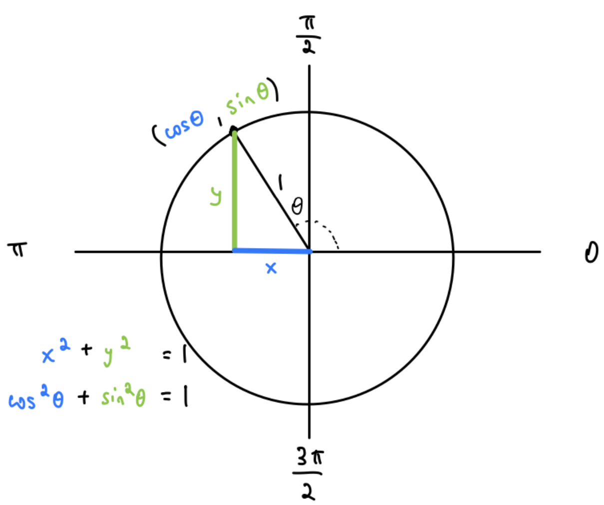




Figure 8: Setups for proving the above identities

# 7 Complex numbers

The general form of a complex number, usually denoted $z$, is $z = a + bi$, $a, b \in \mathbb{R}$.

Note the set of real numbers is a subset of the set of complex numbers: $\mathbb{R} \subset \mathbb{C}$. In general, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

## 7.1 Addition and subtraction

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{7.1.1}$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{7.1.2}$$

## 7.2 Multiplication

Note that (7.2.1) should not be memorized. Simply apply the distributive property and use that $i^2 = (\sqrt{-1})^2 = -1$. Also, if given numbers (not variables), you can and probably should use your TI calculator [13] to evaluate the multiplication.

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{7.2.1}$$

## 7.3 Complex plane

Complex numbers can be graphed on the complex plane, which is just the Cartesian plane but with imaginary numbers on the vertical axis. See Figure 9. This definition allows us to compute distances between complex numbers on the complex plane using the distance formula. Of course, the midpoint formula also holds.

Though it's not tested, see [Wei22a] for an explanation of one of the most beautiful equations in math, $e^{i\pi} + 1 = 0$.

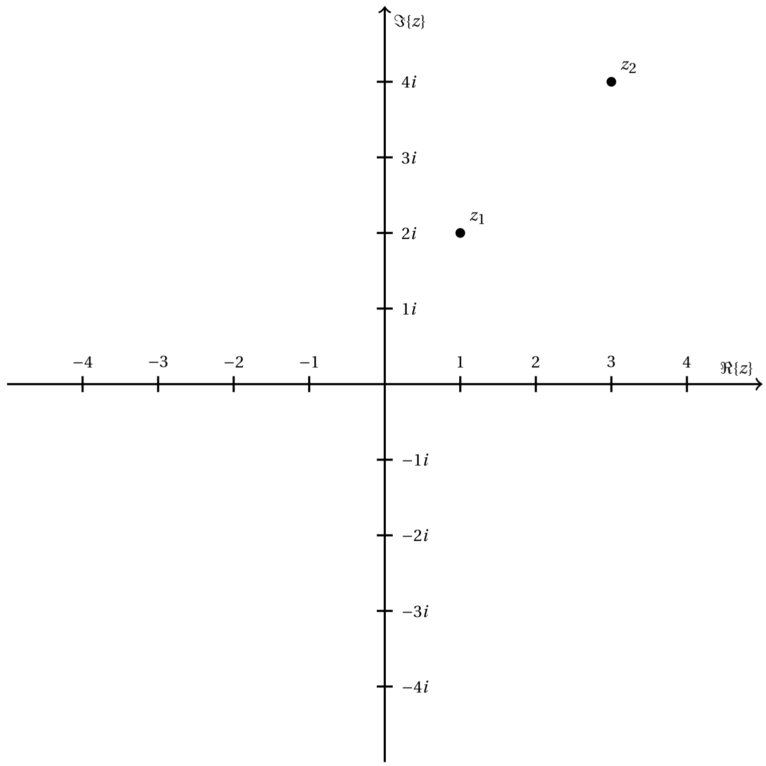



Figure 9: Complex plane with $z_1 = 1 + 2i$ and $z_2 = 3 + 4i$

[13] $i$ can be typed by pressing `2nd` and then `.` (on the numpad)

**Problem 7.3.1: Distance on complex plane**

Compute the distance between $z_1$ and $z_2$ in Figure 9 above. Also, compute the midpoint of $z_1$ and $z_2$.

*Solution.* The distance is $\sqrt{(3-1)^2+(4-2)^2} = \sqrt{8}$. The midpoint is $\frac{1+3}{2} + \frac{2+4}{2}i = 2+3i$. ■

### 7.4 Conjugate

The conjugate of $a+bi$ is $a-bi$. Also, note that $(a+bi)(a-bi) = a^2+b^2$. Similar to $a^2-b^2 = (a-b)(a+b)$, multiplying a complex number by its conjugate cancels terms with $i$ in the result, so the result becomes completely real-valued. This is useful for simplifying expressions with a complex number in the denominator.

## 8 Sequences

A sequence has form $\{a, b, c\ldots\}$ A partial sum is the sum of part of a sequence. A series is the sum of an infinite sequence.

### 8.1 Arithmetic

A finite arithmetic sequence with $n$ terms is of the form $\{a_1, a_1+d, a_1+2d, \ldots, a_1+(n-1)d\}$, where $d$ is the common difference. So, the $n^{\text{th}}$ term is given by $a_1+(n-1)d$.

The sum of a finite arithmetic sequence is given by the sum of the first and last term multiplied by half of the number of terms, [14] as shown in (8.1.1).

$$\frac{n}{2}(\text{first}+\text{last}) = \frac{n}{2}(a_1+a_1+(n-1)d) \tag{8.1.1}$$

### 8.2 Geometric

A geometric sequence is of the form $\{a_1, a_1 r, a_1 r^2\ldots\}$, where $r$ is the common ratio. So, the $n^{\text{th}}$ term is given by $a_1(d^{n-1})$. There are formulas for the sums of finite and infinite geometric sequences, [15] but you don't need to memorize them for the ACT®.

### 8.3 Recursive

I will approach recursive formulas for arithmetic/geometric sequences by example. Consider the recursive sequence defined by $a_1 = 10$ and $a_{n+1} = 2a_n$. Following these two rules, $a_2 = 20$, $a_3 = 40\ldots$, which is clearly a geometric sequence. Using Section 8.2, the explicit formula must be of the form $a_n = a_1(d^{n-1})$. Here, the explicit formula [16] is given by $a_n = 10(2^{n-1})$.

However, note that if the first term is defined as $a_0 = 10$, then the explicit formula [17] would need to be $a_n = 10(2^n)$.

Lastly, note that whereas the explicit formula requires one evaluation to get any term, the recursive formula needs to be applied several times to reach an arbitrary term. [18] That is, the recursive formula describes how to get to term $n+1$ if you currently have the value of term $n$. It should make sense, then, that this type of recursive formula involving multiplication has an explicit formula involving exponentiation (geometric). Likewise, this type of recursive formula involving addition has an explicit formula involving multiplication (arithmetic).

[14] Consider $\{1,4,7,10,13,16\}$. 1+16=4+13=7+10=17. Each pair has the same sum, so to get the total sum, we add the first and last terms and multiply by the number of pairs

[15] An infinite geometric series converges if $r < 1$

[16] You should verify that this formula works

[17] Again, verify

[18] Unless you were looking for the first or second term

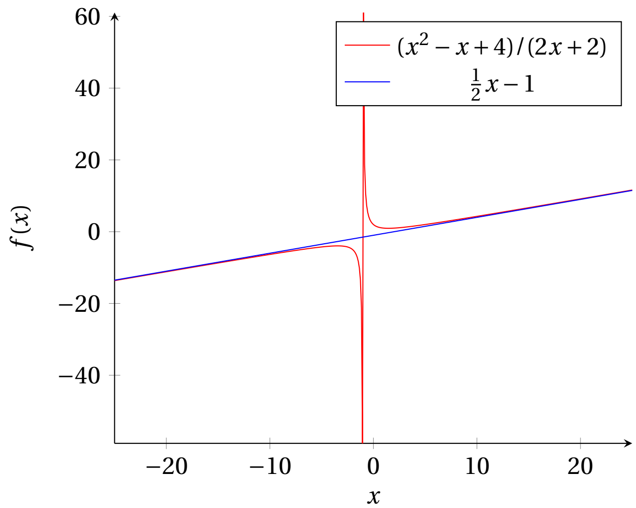



Figure 10: Oblique asymptote

# 9 Rational functions

A rational function is a ratio of two polynomials. [19] $f$ is a rational function if it can be written in the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are both polynomials and $Q(x)$ is not 0.

## 9.1 Oblique asymptote

A rational function $f(x) = \frac{P(x)}{Q(x)}$ has an oblique (slanted) asymptote if the degree of its numerator is *exactly one more than* the degree of its denominator. That is, $\deg P(x) = 1 + \deg Q(x)$. If so, then the (linear) equation of the asymptote is given by the quotient of $P(x)/Q(x)$, using polynomial long division. The remainder can be ignored. See Figure 10, noting that the quotient (remainder ignored) of $(x^2 - x + 4)/(2x + 2)$ is $\frac{1}{2}x - 1$. This can be computed by the polynomial long division method shown in this MathIsFun article.

In Figure 11, I use `Mathematica`, which is the most useful tool for mathematics that I know of. The software costs money, but it may be offered for free at university.

```
In[1]:= PolynomialQuotient[x^2 - x + 4, 2 x + 2, x]

Out[1]= -1 + x/2
```

Figure 11: Polynomial long division using `Mathematica`

[19] This is analogous to a rational number being a ratio of two integers $\frac{p}{q}$, $q \neq 0$

# 10 Number theory

The rest of this document has some advanced notation and is written somewhat rigorously, like math textbooks are. Note that although these concepts are tested on the ACT®, the notation, formal definitions, and proofs are definitely not.

However, if you plan on majoring in computer science, mathematics, physics, etc., you may see the rest of this as a (hopefully) gentle introduction to college-level proof-based mathematics.

## 10.1 Basic properties

Here are some basic properties of integers that you should know.

| | |
|---|---|
| even + even | even |
| even + odd | odd |
| odd + odd | even |
| even × even | even |
| even × odd | even |
| odd × odd | odd |
| negative$^{\text{even}}$ | positive |
| negative$^{\text{odd}}$ | negative |

Figure 12: Basic properties of elements of $\mathbb{Z}$

These do not need to be memorized. You can simply derive any one of these from a simple example if you forget. [20] For completeness, I'll show short proofs of two of these after introducing the necessary notation.

### 10.1.1 Notation

| | |
|---|---|
| $\exists$ | there exists |
| $\in$ | in |
| $\mathbb{Z}$ | the set of integers $\{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$ |
| $\implies$ | implies |

Figure 13: Notation

For example, the following are true definitions of even and odd numbers and will be used in the below proofs: $a$ is even $\implies \exists m \in \mathbb{Z}$ such that $a = 2m$. $b$ is odd $\implies \exists n \in \mathbb{Z}$ such that $b = 2n + 1$.

### 10.1.2 Proofs

You will never need to prove anything for the ACT®, a multiple choice test, but I want to introduce proofs somewhere in this document :)

**Problem 10.1.1: even × even**

Show that if $a$ and $b$ are even, then $ab$ is even.

*Solution.* $a$ is even $\implies \exists m \in \mathbb{Z}$ such that $a = 2m$. Similarly, $b$ is even $\implies \exists n \in \mathbb{Z}$ such that $b = 2n$. Then $ab = 4mn = 2(2mn) \implies ab$ is even. ■

**Problem 10.1.2: odd × odd**

Show that if $a$ and $b$ are odd, then $ab$ is odd.

[20] For example, if we forget the result of odd × odd, then we consider 3 × 3, for which the result is odd

*Solution.* $a$ is odd $\implies \exists m \in \mathbb{Z}$ such that $a = 2m+1$. Similarly, $b$ is odd $\implies \exists n \in \mathbb{Z}$ such that $b = 2n+1$. Then $ab = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1 \implies ab$ is odd. ■

## 10.2 Primes

For further clarification about this topic, [21] see [Ros18] section 4.3.

**Definition 10.2.1: Primes**

A prime is a natural number with only two factors, 1 and itself. [a] The primes are $\{2,3,5,7,11\ldots\}$. A composite number is a number that is not prime.

[a] Note this implies 1 is not prime

### 10.2.1 Prime factorization

**Theorem 10.2.1: Fundamental theorem of arithmetic**

Every integer greater than 1 can be represented uniquely as a product of prime numbers. In other words, given any integer $n > 1$, there exist unique positive integer $k$, distinct prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ such that

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_k^{e_k} \tag{10.2.1}$$

This fact is useful for finding the greatest common factor and least common denominator.

### 10.2.2 Greatest common divisor

**Definition 10.2.2: GCD**

$\gcd(a,b)$ is the greatest number that is a factor of two numbers $a$ and $b$.

By Definition 10.2.2, $\gcd(a,b)$ is the product of the common factors in the factorizations of $\{a,b,c\ldots\}$. [22] Suppose the prime factorizations of $a$ and $b$ are $a = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ and $a = p_1^{b_1} p_2^{b_2} \ldots p_n^{b_n}$. [23] Then

$$\gcd(a,b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \ldots p_n^{\min(a_n,b_n)} \tag{10.2.2}$$

And this definition is easily extended to $\gcd(a,b,c,\ldots)$.

### 10.2.3 Least common multiple

**Definition 10.2.3: LCM**

$\operatorname{lcm}(a,b)$ is the smallest number that is a multiple of two numbers $a$ and $b$.

Using the same definitions of $a$ and $b$ above, $\operatorname{lcm}(a,b)$ is

$$\operatorname{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \ldots p_n^{\max(a_n,b_n)} \tag{10.2.3}$$

[21] And also literally anything else in discrete math, which I attempt to gently introduce through this document. This was the textbook for my own MATH 381 course, taught by Chad Kelterborn (now a PhD student at Johns Hopkins). MATH 381 set me up for success in both of my undergraduate degrees. The author of the textbook is formerly from AT&T Laboratories, which came from Bell Labs, which is where `Unix`, `C`, `C++` and so much more are from. Legendary!

[22] You should verify that this makes sense. You could also read [Ros18] page 281

[23] All primes occurring in the prime factorization of $a$ or $b$ are included in both factorizations, with zero exponents if necessary

## 10.3 Modular arithmetic

**Definition 10.3.1: Binary mod operator**

$a \bmod d$ is defined as the remainder of $a$ divided by $d$.

More formally, here is the division algorithm.[24]

**Theorem 10.3.1: Division algorithm**

Let $a$ be an integer and $d$ a positive integer. Then there exist unique integers $q$ and $r$, with $0 \le r < d$, such that $a = dq + r$.

Here, $d$ is the divisor, $q$ is the quotient $\lfloor a/d \rfloor$[25] and $r$ is the remainder (which can be 0 but can't be equal to $d$). Modular arithmetic is useful for things that repeat cyclically. For example,

**Problem 10.3.1: Repeating digits**

What is the 322$^{\text{nd}}$ digit after the decimal point in the repeating decimal $0.\overline{1357}$?

*Solution.* Since the cycle is of length 4, we compute $322 \bmod 4 = 2$. Without using a modulus function on the calculator, this can be calculated by $322 - 4\lfloor \frac{322}{4} \rfloor = 2$. [26] Now we find the 2$^{\text{nd}}$ number [27] in the cycle, which is 3. ■

**Problem 10.3.2: Powers of $i$**

Where $i = \sqrt{-1}$, what is $i^{799}$?

*Solution.* The first four powers of $i$, starting from 0, are $\{1, i, -1, -i\}$, and the next power is 1, so we indeed have another cycle of length 4. To calculate $i^{799}$, we compute $799 \bmod 4 = 3$. Since this time we started counting from 0, the power of $i$ that corresponds to remainder 3 is $-i$, so $i^{799} = -i$. ■

# 11 Graph theory

See 2022 June E26 question 45 for a literal graph theory question... this topic most likely won't be asked again for a very long time.

For further details on Euler paths and Euler circuits, please see these slides, which prove the ideas shown below.

Anyway, an Euler path (or trail) of a graph is a path that visits each edge once. The first and last vertices of the path are different. Figure 14 shows an example.

For an Euler path to exist, the degree (number of edges coming out of) each node must be even, except for (and only) the start and end nodes of the path, which have odd degree. In other words, exactly two nodes have odd degree. In Figure 14, note that B and C both have odd degree.

An Euler circuit is similar, but the start and end nodes of the path are the same, so the degree of each node must be even. See Figure 15, and note that the graph is slightly different from the previous one.

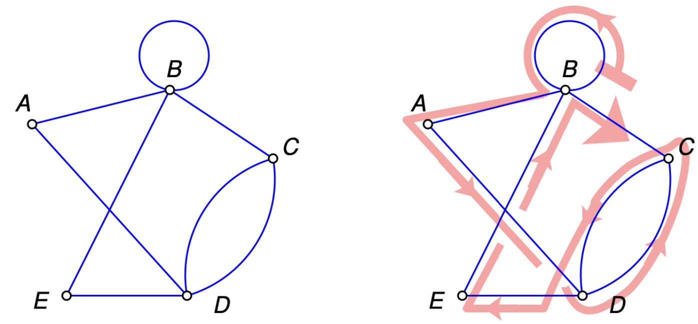
[24] It isn't really an algorithm

[25] $\lfloor x \rfloor$, pronounced "the floor of $x$," denotes the largest integer less than or equal to $x$. For example, $\lfloor 1 \rfloor = \lfloor \sqrt{2} \rfloor = \lfloor 1.6\bar{9} \rfloor = 1$. Think of it as rounding $x$ down or ignoring its decimal digits

[26] This uses $r = a - dq$ from the Division algorithm

[27] Note that the 1$^{\text{st}}$ number corresponds to remainder 1, so the 2$^{\text{nd}}$ number corresponds to remainder 2, as desired. The 4$^{\text{th}}$ number corresponds to remainder 0.

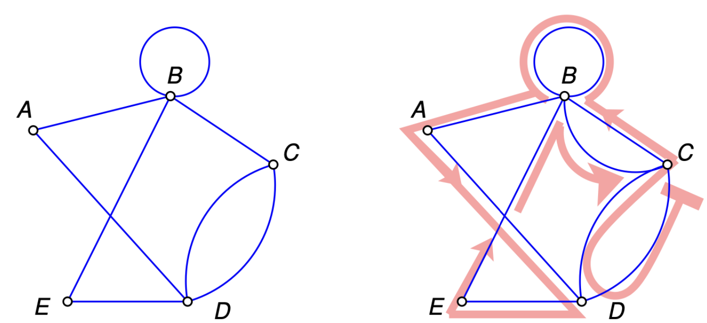
## Euler Paths and Euler Circuits




**An Euler path: BBADCDEBC**

Figure 14: Euler path

## Euler Paths and Euler Circuits




**An Euler circuit: CDCBBADEBC**

Figure 15: Euler circuit

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